JOURNAL OF MULTIVARIATE ANALYSIS 6, 455–471 (1976)

On α-Factors of Multivariate Infinitely Divisible Probabilities

ROGER CUPPENS

Université Paul Sabatier, Toulouse, France

Communicated by E. Lukacs

Let p be an infinitely divisible n-variate probability having μ for Poisson measure. We give here some sufficient conditions for p to belong to the class I_α^n of n-variate probabilities having only infinitely divisible α-factors. These results are interesting since they are concerned with the case when μ is a continuous measure.

1. INTRODUCTION

The definitions and notations are those of the author's monograph [2]. We recall the

DEFINITION [2, p. 199]. If the characteristic function $\hat{p}$ of an n-variate probability p has no real zeros, we say that an n-variate probability q is an α-factor of p if there exist some n-variate probabilities $r_1, \ldots, r_m$ and some positive constants $\alpha, \beta_1, \ldots, \beta_m$ such that

$$\hat{p} = \hat{q}^\alpha \prod_{j=1}^{m} (\hat{r}_j)^{\beta_j}.$$

We denote by I_α^n the set of n-variate probabilities having only infinitely divisible α-factors. Clearly, I_α^n is a subset of the set I_0^n of n-variate probabilities having only infinitely divisible factors. We do not know at present if $I_\alpha^n \neq I_0^n$.

We gave briefly in [2, Chap. 9] some results on the description of I_α^n. Using a result by Čistjakov [1] and a new method due to Fryntov [3] in the case $n = 1$, we give here some new results on the class I_α^n. These results are

Received

AMS 1970 subject classification: Primary 60E05.

Key words and phrases: Infinitely divisible, characteristic functions, Poisson measure, α-factors.

interesting since they are concerned with infinitely divisible probabilities having a continuous Poisson measure. Moreover, we return to [2, Theorems 9.3.4 and 9.3.5] since their proof seems to be too concise.

2. An Analytical Property of α-Factors

Let $\Gamma \subset R^n$ be a convex set containing the origin. An n-variate probability p belongs to $\mathscr{A}_\Gamma$ if $\int e^{-(y,x)}p(dx)$ converges for any $y \in \Gamma$. In this case, we can define

$$\hat{p}(t) = \int e^{i(t,x)}p(dx)$$

for any $t \in R^n + i\Gamma$.

Theorem 1. *Let $\Gamma \subset R^n$ be a convex set containing the origin, p_j be an n-variate probability ($j = 1,..., m$) and α_j be a positive constant ($j = 1,..., m$). If the n-variate probability p belongs to $\mathscr{A}_\Gamma$ and if there exists some $\rho > 0$ such that the relation*

$$\hat{p}(t) = \prod_{j=1}^{m} (\hat{p}_j(t))^{\alpha_j} \tag{1}$$

holds for any $t \in R^n$, $\| t \| < \rho$, then p_j belongs to $\mathscr{A}_\Gamma$ ($j = 1,..., m$) and the relation (1) holds for any $t \in i\Gamma$. Moreover, if $\hat{p}$ does not vanish in $R^n + i\Gamma$, then $\hat{p}_j$ does not vanish in $R^n + i\Gamma$ ($j = 1,..., m$) and the relation (1) holds for any $t \in R^n + i\Gamma$.

Proof. In the case $n = 1$, this result is due to Čistjakov [1]. The general case follows from the case $n = 1$ by the usual projection method (cf. [2, Theorem 5.3.1]).

From this result and from the ridge property, we can deduce the

Corollary 1. *With the conditions of the preceding theorem, we have*

$$| \hat{p}(x + iy)/\hat{p}(iy)|^{1/\alpha_j} \leqslant | \hat{p}_j(x + iy)/\hat{p}_j(iy)| \leqslant 1$$

for any $x \in R^n$ and $y \in \Gamma$ ($j = 1,..., m$). Moreover, if $\hat{p}$ has no zeros in $R^n + i\Gamma$, we have

$$\begin{aligned} 0 &\leqslant \operatorname{Re}[\log \hat{p}_j(iy) - \log \hat{p}_j(x + iy)] \\ &\leqslant (1/\alpha_j) \operatorname{Re}[\log \hat{p}(iy) - \log \hat{p}(x + iy)] \end{aligned}$$

for any $x \in R^n$ and $y \in \Gamma$ ($j = 1,..., m$).

From Theorem 1 we also deduce, with the same method as for ordinary decompositions (cf. [2, pp. 114–116]), COROLLARY 2. *With the conditions of Theorem* 1, *if* p *is bounded in the direction* θ, *then* p_j *is bounded in the direction* θ $(j = 1,..., m)$ *and*

$$\operatorname{ext}_\theta p = \sum_{j=1}^{m} \alpha_j \operatorname{ext}_\theta p_j .$$

COROLLARY 3. *Let the measure* μ *be bounded in the direction* θ *and let* ν *be a bounded* n*-variate signed measure,* $k = \exp(-\mu(R^n))$, $l = \exp(-\nu(R^n))$. *If* $q = l \exp \nu$ *is an* α*-factor of* $p = k \exp \mu$, *then* ν *is bounded in the direction* θ *and*

$$\operatorname{ext}_\theta \nu \leqslant \sup(0, \operatorname{ext}_\theta \mu).$$

3. RELATIONS BETWEEN SUPPORTS

It is well known that if an n-variate probability q divides an n-variate probability p, there exists some $m \in R^n$ such that $S(q) \subset S(p) + m$. One of the main difficulties of the problems of α-decompositions is the lack of such a simple relation between the supports of α-factors. We know only a result when the supports are enumerable [2, Theorem 9.1.1]. Using Fyntov's method [3], we prove now the

LEMMA 1. *Let* $A_k(\theta)$ *be the half-space defined by*

$$\begin{aligned} A_k(\theta) &= \{x \in R^n \colon (x, \theta) \geqslant k\} \qquad \text{if} \quad k > 0, \\ &= \{x \in R^n \colon (x, \theta) > 0\} \qquad \text{if} \quad k = 0, \end{aligned}$$

and p *be an* n*-variate probability concentrated on* $\{0\} \cup A_k(\theta)$ *for some* $k \geqslant 0$ *and some* $\theta \in R^n$ *such that* $p(\{0\}) \neq 0$. *If*

$$\hat{p} = \prod_{j=1}^{m} (\hat{p}_j)^{\alpha_j} \tag{2}$$

where p_j *is an* n*-variate probability and* α_j *is a positive constant* $(j = 1,..., m)$ *and if* $\hat{p}$ *does not vanish in* $R^n + i\Gamma$ $(\Gamma = \{\lambda\theta \colon \lambda \geqslant 0\})$, *there exists some* $b_j \in R^n$ $(j = 1,..., m)$ *such that*

(a) $\sum_{j=1}^{m} \alpha_j b_j = 0$,

(b) $p_j * \delta_{-b_j}$ *is concentrated on* $\{0\} \cup A_k(\theta)$,

(c) $\prod_{j=1}^{m} p_j(\{b_j\})^{\alpha_j} = p(\{0\})$.

Proof. From Corollary 2, we deduce the existence of $m_j \in R$ such that

$$\sum_{j=1}^{m} \alpha_j m_j = 0$$

and $q_j = p_j * \delta_{-m_j\theta}$ is concentrated on $\{x\colon (x, \theta) \geqslant 0\}$ $(j = 1,\ldots, m)$. Let $q_j = \rho_j + \sigma_j$ where

$$\rho_j(B) = q_j(B \cap \theta^{\perp}),$$
$$\sigma_j(B) = q_j(B \cap (\theta^{\perp})^c)$$

for any Borel set B. Since p belongs to $\mathscr{A}_\Gamma$, it follows from Theorem 1 that the relation

$$\hat{p}(t + iy\theta) = \prod_{j=1}^{m} (\hat{q}_j(t + iy\theta))^{\alpha_j}$$

holds for any $y \geqslant 0$ and $t \in R^n$. Since

$$\lim_{y\to+\infty} \hat{p}(t + iy\theta) = p(\{0\}),$$
$$\lim_{y\to+\infty} \hat{q}_j(t + iy\theta) = \hat{\rho}_j(t),$$

we have

$$\prod_{j=1}^{m} (\hat{\rho}_j(t))^{\alpha_j} = p(\{0\}).$$

Since the α-factors of degenerate probabilities are degenerate probabilities, we have $\rho_j = k_j\delta_{a_j}$, k_j being some positive constant and $a_j \in R^n$ $(j = 1,\ldots, m)$ and this implies the validity of (a) and (c) with $b_j = a_j + m_j\theta$ $(j = 1,\ldots, m)$. If $k = 0$, the theorem is proved.

We suppose now $k > 0$. If $r_j = p_j * \delta_{-b_j}$ $(j = 1,\ldots, m)$, we have

$$\hat{p}(t + iy\theta) = \prod_{j=1}^{m} (\hat{r}_j(t + iy\theta))^{\alpha_j}$$

for any $y \geqslant 0$ and $t \in R^n$. Since

$$\hat{p}(iy\theta) \leqslant p(\{0\}) + [1 - p(\{0\})]\, e^{-yk},$$
$$\hat{r}_j(iy\theta) \geqslant r_j(\{0\}) + r_j(\{x\colon 0 < (x, \theta) \leqslant l\})\, e^{-yl},$$

we have the inequality

$$\prod_{j=1}^{m} [r_j(\{0\}) + r_j(\{x : 0 < (x, \theta) \leqslant l)\, e^{-yl}]^{\alpha_j}$$
$$\leqslant p(\{0\}) + [1 - p(\{0\})]\, e^{-yk}.$$

If we let $\omega = [1 - p(\{0\})]/p(\{0\})$, $\omega_j = r_j(\{x\colon 0 < (x, \theta) \leqslant l\})/r_j(\{0\})$ $(j = 1,\ldots, m)$ and if we use (c), we obtain

$$\prod_{j=1}^{m} (1 + \omega_j e^{-yl})^{\alpha_j} \leqslant 1 + \omega e^{-yk}$$

and therefore

$$(1 + \omega_j e^{-yl})^{\alpha_j} \leqslant 1 + \omega e^{-yk}, \qquad (j = 1,\ldots, m). \tag{3}$$

Since, when $y \to +\infty$,

$$(1 + \omega_j e^{-yl})^{\alpha_j} = 1 + (\alpha_j \omega_j + o(1))\, e^{-yl},$$

this implies $\omega_j = 0$ (that is, $r_j(\{x\colon 0 < (x, \theta) \leqslant l\}) = 0$) for any $l < k$. Therefore, we obtain the validity of (b).

4. The Main Results

Lemma 2. *Let $A \in F_\sigma^n$ be a set contained in the half-space $\{x \in R^n\colon (x, \theta) > 0\}$ for some $\theta \in R^n$, p_j be an n-variate probability, α_j be a constant greater than one $(j = 1,\ldots, m)$ and $\tau = \sup_{1 \leqslant j \leqslant m} \alpha_j$. If $p = \omega \exp \mu$ where μ is a bounded measure on A and satisfying*

$$\omega = \exp(-\mu(R^n)) > (1 + 2^{-m\tau})^{-1} \tag{4}$$

and if (2) is satisfied, there exist some $b_j \in R^n$ and some $\nu_j \in \mathcal{M}_n$ $(j = 1,\ldots, m)$ satisfying the following conditions

(a) $\sum_{j=1}^{m} \alpha_j b_j = 0$;

(b) $p_j = \omega_j(\delta_{b_j} * \exp \nu_j)$, $\omega_j = \exp(-\nu_j(R^n))$ $(j = 1,\ldots, m)$;

(c) *ν_j is concentrated on $C(\mu) \cap (N)A$;*

(d) *$\nu_j(B) \geqslant 0$ for any Borel set B satisfying*

$$B \cap \left(\bigcup_{k=2}^{\infty} (k)A\right) = \varnothing. \tag{5}$$

Proof. From Lemma 1, we deduce the existence of some $b_j \in R^n$ $(j = 1,\ldots, m)$ such that

$$\sum_{j=1}^{m} \alpha_j b_j = 0$$

and such that $q_j = p_j * \delta_{-b_j}$ is concentrated on $\{x: (x, \theta) > 0\}$ and satisfies

$$\prod_{j=1}^{m} (q_j(\{0\}))^{\alpha_j} = p(\{0\}). \tag{6}$$

From (4), we deduce that

$$\omega = p(\{0\}) > \tfrac{1}{2}$$

and, since $\alpha_j \geqslant 1$ ($j = 1,..., m$), we have from (6)

$$q_j(\{0\}) > \tfrac{1}{2}$$

($j = 1,..., m$). From [2, Theorem 4.3.7], we deduce the existence of $\nu_j \in \mathscr{M}_n$ satisfying

$$q_j = \omega_j \exp \nu_j\,, \qquad (j = 1,..., m)$$

and

$$\omega_j = q_j(\{0\}) = \exp(-\nu_j(R^n)) > \tfrac{1}{2}\,.$$

It remains to show that ν_j satisfies conditions (c) and (d).

Using Fryntov's method [3], we show first that ν_j is concentrated on $(N)A$. Let

$$q_j = \omega_j(\delta_0 + \sigma_j),$$
$$p^{*d} = \omega^d(\delta_0 + \rho_d), \qquad (d \in N).$$

Clearly, $\| \sigma_j \| < 1$ and ρ_d is concentrated on $(N)A$. From (6) and (2), it follows that

$$\prod_{j=1}^{m} (1 + \hat{\sigma}_j(t))^{\beta_j} = 1 + \hat{\rho}_d(t)$$

where $\beta_j = d\alpha_j$. The expansion of $(1 + z)^{\beta_j}$ in Taylor series gives

$$(1 + \hat{\sigma}_j(t))^{\beta_j} = 1 + \sum_{k=1}^{\infty} a_{j,k}(d)(\hat{\sigma}_j(t))^k = 1 + \hat{r}_{j,d}(t)$$

where $r_{j,d}$ is the signed measure defined by

$$r_{j,d} = \sum_{k=1}^{\infty} a_{j,k}(d)\, \sigma_j^{*k}$$

and

$$a_{j,k}(d) = \frac{\beta_j(\beta_j - 1) \cdots (\beta_j - k + 1)}{k!}\,.$$

Since $a_{j,k}(d) \geqslant 0$ if $k \leqslant |\beta_j| + 1$ and $-1 < a_{j,k}(d) < 0$ if $k > |\beta_j| + 1$, we have

$$\sum_{k=|\beta_j|+2}^{\infty} |a_{j,k}(d)| \, \|\sigma_j\|^k \leqslant \sum_{k=|\beta_j|+2}^{\infty} \|\sigma_j\|^k \leqslant K^d/(1-K) \tag{7}$$

where $K = \sup_{1\leqslant j\leqslant m} \|\sigma_j\|$. Since

$$\|r_{j,d}^-\| \leqslant \sum_{k=|\beta_j|+2}^{\infty} |a_{j,k}(d)| \, \|\sigma_j\|^k,$$

we obtain from (7)

$$\|r_{j,d}^-\| = O(K^d), \qquad (d \to +\infty). \tag{8}$$

Likewise from

$$\begin{aligned}\|r_{j,d} + \delta_0\| &\leqslant \sum_{k=0}^{\infty} |a_{j,k}(d)| \, \|\sigma_j\|^k \\ &= \sum_{k=0}^{\infty} a_{j,k}(d) \, \|\sigma_j\|^k + 2 \sum_{k=|\beta_j|+2}^{\infty} |a_{j,k}(d)| \, \|\sigma_j\|^k \\ &\leqslant (1+K)^{\beta_j} + (2K^d/(1-K)),\end{aligned}$$

it follows that

$$\|r_{j,d} + \delta_0\| = O[(1+K)^{\tau d}], \qquad (d \to +\infty) \tag{9}$$

and

$$\|r_{j,d}\| = O[(1+K)^{\tau d}], \qquad (d \to +\infty). \tag{10}$$

If we define $s_{j,d}$ by

$$\prod_{\substack{1\leqslant k\leqslant m\\ k\neq j}} (1 + \hat{r}_{k,d}(t)) = 1 + \hat{s}_{j,d}(t)$$

we have

$$\begin{aligned}\|s_{j,d}\| &\leqslant \|s_{j,d} + \delta_0\| \\ &\leqslant \prod_{\substack{1\leqslant k\leqslant m\\ k\neq j}} \|r_{j,d} + \delta_0\| = O[(1+K)^{\tau d(m-1)}], \qquad (d \to +\infty).\end{aligned} \tag{11}$$

It is easily shown by induction on m that

$$\|s_{j,d}^-\| \leqslant (m-1) \sup_{k\neq j} \|r_{j,d}^-\| \, (\sup_{k\neq j} \|r_{k,d} + \delta_0\|)^{m-2}$$

and, therefore, from (8) and (9)

$$\| s_{j,d}^{-} \| = O[K^d(1+K)^{\tau d(m-2)}], \qquad (d \to +\infty). \tag{12}$$

On the other hand, from

$$\|(r_{j,d} * s_{j,d})^{-}\| \leqslant \| r_{j,d}^{-} \| \, \| s_{j,d} \| + \| r_{j,d} \| \, \| s_{j,d}^{-} \|$$

and, from (8), and (10)–(12), we obtain

$$\|(r_{j,d} * s_{j,d})^{-}\| = O[K^d(1+K)^{\tau dm}], \qquad (d \to +\infty). \tag{13}$$

If B is a Borel set satisfying $B \cap (N)A = \varnothing$, since ρ_d is concentrated on $(N)A$, we have $\rho_d(B) = 0$. From

$$\rho_d = r_{j,d} + s_{j,d} + (r_{j,d} * s_{j,d}),$$

it follows that

$$\begin{aligned} [\| r_{j,d}^{-} \| + r_{j,d}(B)] &+ [\| s_{j,d}^{-} \| + s_{j,d}(B)] \\ &+ [\|(r_{j,d} * s_{j,d})^{-}\| + (r_{j,d} * s_{j,d})(B)] \\ &= \| r_{j,d}^{-} \| + \| s_{j,d}^{-} \| + \|(r_{j,d} * s_{j,d})^{-}\|. \end{aligned}$$

Since the three terms between square brackets in the left side of the preceding equality are nonnegative, we have

$$-\| r_{j,d}^{-} \| \leqslant r_{j,d}(B) \leqslant \| s_{j,d}^{-} \| + \|(r_{j,d} * s_{j,d})^{-}\|$$

and this implies with (8), (12) and (13)

$$r_{j,d}(B) = O[K^d(1+K)^{\tau dm}], \qquad (d \to +\infty).$$

From this last relation and (7), we deduce that

$$\sum_{k=1}^{|\beta_j|+1} a_{j,k}(d)\, \sigma_j^{*k}(B) = O[K^d(1+K)^{\tau dm}], \qquad (d \to +\infty)$$

and since

$$0 \leqslant \beta_j \sigma_j(B) \leqslant \sum_{k=1}^{|\beta_j|+1} a_{j,k}(d)\, \sigma_j^{*k}(B),$$

we have

$$\sigma_j(B) = O[K^d(1+K)^{\tau dm}], \qquad (d \to +\infty).$$

From (4)

$$K = \sup_{1 \leqslant j \leqslant m} \frac{1-\omega_j}{\omega_j} \leqslant \frac{1-\omega}{\omega} < 2^{-m},$$

so that

$$K(1+K)^{rm} < 1$$

and, letting $d \to +\infty$, we obtain $\sigma_j(B) = 0$. In other words, σ_j is concentrated on $(N)A$ and, from [2, Theorem 4.3.7], we deduce that ν_j is also concentrated on $(N)A$.

We show now that ν_j is concentrated on $C(\mu)$. Let h be the support function of $C(\mu)$. If, for some $\theta \in R^n$, $h(\theta) \geqslant 0$, we deduce from Corollary 3 that

$$\operatorname{ext}_\theta \nu \leqslant h(\theta). \tag{14}$$

If $h(\theta) < 0$, then

$$A \subset S = \{x \in R^n \colon -(x, \theta) \geqslant -h(\theta)\}$$

and

$$\bigcup_{k=1}^{\infty} (k)A \subset S$$

so that (14) holds also in this case. Since

$$C(\mu) = \bigcap_{\theta \in R^n \setminus \{0\}} \{x \in R^n : (x, \theta) \leqslant h(\theta)\},$$

from the definition of the convex support $C(\nu)$ of ν, we have

$$C(\nu) \subset C(\mu)$$

and ν is concentrated on $C(\mu)$. Finally for any Borel set B satisfying (5)

$$\nu(B) = q(B) \geqslant 0$$

and the lemma is proved.

From this lemma, we deduce the following theorem which extends to $I_\alpha{}^n$ [2, Theorem 7.1.4]:

THEOREM 2. *Let A be an open set satisfying*

$$A^* \cap \left(\bigcup_{k=2}^{\infty} (k)A \right) = \varnothing \tag{15}$$

where A^* is the convex hull of A and let μ be a bounded measure concentrated on A. If $p = \omega \exp \mu$ $[\omega = \exp(-\mu(R^n))]$, then p belongs to I_α^n.

Proof. Exactly as for [2, Theorem 7.1.4], we deduce from (15) the existence of some $\theta \in R^n$ such that $A^* \subset \{x \in R^n: (x, \theta) > 0\}$. Let p_j be some n-variate probability and α_j be some positive constant $(j = 1,\dots, m)$ satisfying (2). If $l \in N$ satisfies $l\alpha_j = \beta_j \geqslant 1$ $(j = 1,\dots, m)$ and if $q = p^{*l} = \omega^l \exp(l\mu)$, we have

$$\hat{q} = \prod_{j=1}^{m} (\hat{p}_j)^{\beta_j}.$$

If $\Gamma = \{\lambda\theta: \lambda \geqslant 0\}$, since μ is concentrated on A, q belongs to $\mathscr{A}_\Gamma$ and, from Theorem 1, the relation

$$\hat{q}(t + iy\theta) = \prod_{j=1}^{m} (\hat{p}_j(t + iy\theta))^{\beta_j}$$

holds for any $y \geqslant 0$ and $t \in R^n$.

Let $y_0 \geqslant 0$ satisfy

$$\omega' = \exp(-\nu(R^n)) > (1 + 2^{-m\tau})^{-1}$$

where $\tau = \sup_{1 \leqslant j \leqslant m} \beta_j$ and ν is the measure defined by

$$\nu(B) = l \int_B \exp(-(x, y_0\theta))\, \mu(dx)$$

for any Borel set B. Clearly ν is concentrated on A. If

$$\hat{r}(t) = [\hat{q}(iy_0\theta)]^{-1}\, \hat{q}(t + iy_0\theta)$$
$$\hat{r}_j(t) = [\hat{p}_j(iy_0\theta)]^{-1}\, \hat{p}_j(t + iy_0\theta),$$

we have

$$\hat{r} = \prod_{j=1}^{m} (\hat{r}_j)^{\beta_j}$$

and from [2, Theorem 5.2.2], we deduce that $r = \omega' \exp \nu$.

The conditions of Lemma 2 are satisfied by r and r_j and we obtain the existence of $b_j \in R^n$ and $\nu_j \in \mathscr{M}_n$ satisfying the following conditions

(a) $\sum_{j=1}^{m} \beta_j b_j = 0$;

(b) $r_j = \omega_j(\delta_{b_j} * \exp \nu_j)$, $\omega_j = \exp(-\nu_j(R^n))$;

(c) ν_j is concentrated on $(N)A \cap \overline{A^*}$;

(d) $\nu_j(B) \geqslant 0$ for any Borel set satisfying (5).

Since A^* and $\bigcup_{k=2}^{\infty} (k)A$ are both open sets, if (15) holds, then

$$(N)A \cap \overline{A^*} = A$$

and this implies that ν_j is nonnegative, so that r_j is infinitely divisible. From [2, Theorem 5.2.3], we deduce that p_j is infinitely divisible ($j = 1,..., m$) and p belongs to $I_\alpha{}^n$.

COROLLARY 4. *Let μ be an n-variate absolutely continuous measure so that $A = \{x \in R^n : (d\mu/d\lambda)(x) > 0\}$ is an open set and $\omega = \exp(-\mu(R^n))$. Then $p = \omega \exp \mu$ belongs to $I_\alpha{}^n$ if and only if A satisfies* (15).

The sufficiency follows from the preceding theorem while we deduce the necessity from a result by Mase [5]. In the case $n = 1$, we can suppress the assumption that A is an open set (cf. Mase [4]):

COROLLARY 5. *Let μ be a univariate absolutely continuous measure and $\omega = \exp(-\mu(R))$. Then $p = \omega \exp \mu$ belongs to $I_\alpha{}^1$ if and only if there exists some $a > 0$ such that the set $A = \{x \in R: (d\mu/d\lambda)(x) > 0\}$ is contained in $[a, 2a]$ or $[-2a, -a]$.*

THEOREM 3. *Let $A \in F_\sigma{}^n$ be an independent set contained in a half space $\{x \in R^n: (x, \theta) > 0\}$ for some $\theta \in R^n$. If $p = \omega \exp \mu$ where μ is a bounded measure concentrated on A and $\omega = \exp(-\mu(R^n))$, then p belongs to $I_\alpha{}^n$.*

Proof. Let p_j be an n-variate probability and α_j be a positive constant ($j = 1,..., m$) satisfying (2). If $l \in N$ is such that $l\alpha_j = \beta_j > 1$ ($j = 1,..., m$), we can find $y_0 \geqslant 0$ such that

$$\omega' = \exp(-\nu(R^n)) > (1 + 2^{-m\tau})^{-1}$$

where $\tau = \sup_{1 \leqslant j \leqslant m} \beta_j$ and ν is the measure defined by

$$\nu(B) = l \int_B \exp(-(x, y_0\theta))\, \mu(dx)$$

for any Borel set B. If

$$\hat{q}(t) = [\hat{p}^l(iy_0\theta)]^{-1}\, \hat{p}^l(t + iy_0\theta),$$
$$\hat{q}_j(t) = [\hat{p}_j(iy_0\theta)]^{-1}\, \hat{p}_j(t + iy_0\theta),$$

we have

$$\hat{q} = \prod_{j=1}^{m} (\hat{q}_j)^{\beta_j}$$

and $q = \omega' \exp \nu$. From Lemma 2, we deduce the existence of some $b_j \in R^n$ and some $\nu_j \in \mathscr{M}_n$ satisfying the following conditions:

(a) $\sum_{j=1}^{m} \beta_j b_j = 0$;

(b) $r_j = q_j * \delta_{-b_j} = \omega_j \exp \nu_j$, $\omega_j = \exp(-\nu_j(R^n))$;

(c) ν_j is concentrated on $(N)A$ $(j = 1,..., m)$.

This implies

$$\sum_{j=1}^{m} \beta_j \nu_j = \nu.$$

Let $A' \in F_\sigma^1$ be an independent set contained in $[1, 2[$ and equipotent to A, ϕ be a bimeasurable bijection of A on A' and Φ be the isomorphism from the algebra $\mathscr{M}((N)A)$ on the algebra $\mathscr{M}((N)A')$ induced by ϕ (cf. [2, pp. 160–162]). Since Φ is continuous for the weak convergence, we have

$$\Phi(q) = \omega' \exp(\Phi(\nu)), \qquad \Phi(r_j) = \omega_j \exp(\Phi(\nu_j))$$

and

$$\sum_{j=1}^{m} \beta_j \Phi(\nu_j) = \Phi(\nu),$$

so that

$$\widehat{\Phi(q)} = \prod_{j=1}^{m} (\widehat{\Phi(r_j)})^{\beta_j}.$$

From the preceding theorem, $\Phi(q)$ belongs to I_α^1 and consequently $\Phi(r_j)$ is an infinitely divisible probability $(j = 1,..., m)$. From the properties of Φ, the same holds for r_j and, from [2, Theorem 5.2.3], we deduce that p_j is infinitely divisible. Therefore p belongs to I_α^n.

With the same proof, we can obtain the following extension of Theorem 3 (for the definition, cf. [2, p. 164]):

THEOREM 4. *Let $A \in F_\sigma^n$ be a set contained in a half space $\{x \in R^n: (x, \theta) > 0\}$ for some $\theta \in R^n$ and the projection of which on a subspace of R^n is a generalized independent set of type 2. If $p = \omega \exp \mu$ where μ is a bounded measure concentrated on A and $\omega = \exp(-\mu(R^n))$, then p belongs to I_α^n.*

For ordinary decompositions, similar theorems hold without the assumption that A is contained in a half space (cf. [2, Theorems 7.4.2 and 7.4.3]). For α-decompositions, we do not know if this assumption can be removed except when A is enumerable (cf. Theorem 6 below).

We show now the following lemma (for definitions of Linnik sets and issue, cf. [2, pp. 122, 139]):

LEMMA 3. *Let $\Lambda = \Lambda_1 + \cdots + \Lambda_n$ where $\Lambda_j = \{\lambda_{j,k} e_j\}_{k=0}^{\infty}$, $\{\lambda_{j,k}\}_{k=0}^{\infty}$ being a Linnik set of nonnegative numbers ($\lambda_{j,0} = 0$) and $\{e_1, ..., e_n\}$ being the canonical basis of R^n and let $B = \prod_{j=1}^{\infty} [0, a_j]$, $0 \leqslant a_j < \lambda_{j,1}$. In the conditions of Lemma 2, if μ is concentrated on $\Lambda \cup B$ and satisfies*

$$\mu(\{x \in R^n : \| x \| > u\}) = O(\exp(-Ku^2)) \qquad (u \to +\infty)$$

for some $K > 0$, then ν_j is concentrated on $\Lambda \cup B$ and if $x \in \Lambda$ has an issue with respect to μ, then

$$0 \leqslant \alpha_j \nu_j(\{x\}) \leqslant \mu(\{x\}), \qquad (j = 1, ..., m).$$

Proof. Once the representation $p_j = \omega_j(\delta_{b_j} * \exp \nu_j)$ obtained, the proof given for ordinary decompositions ([2, pp. 152–156]) uses only rige property. Therefore the result holds also for α-decompositions.

From Lemma 3, we deduce the following extension of [2, Theorem 7.1.7] (the proof is the same and can be omitted):

THEOREM 5. *Let $\Lambda = \Lambda_1 + \cdots + \Lambda_n$ where $\Lambda_j = \{\lambda_{j,k} e_k\}_{k=0}^{\infty}$, $\{\lambda_{j,k}\}_{k=0}^{\infty}$ being a Linnik set of nonnegative numbers ($\lambda_{j,0} = 0$) and $\{e_1, ..., e_n\}$ being the canonical basis of R^n. Let $A = \prod_{j=1}^{n} [0, a_j] \cap \{x = (x_1, ..., x_n) \in R^n : \sum_{j=1}^{n} a_j x_j > \frac{1}{2}\}$, $0 \leqslant a_j < \lambda_{j,1}$ ($j = 1, ..., n$). If the bounded n-variate measure μ satisfies the following conditions:*

(a) *μ is concentrated on $\Lambda \cup A$;*

(b) *any point of Λ has an issue with respect to μ;*

(c) *for some $K > 0$,*

$$\mu(\{x \in R^n : \| x \| > u\}) = O(\exp(-Ku^2)) \qquad (u \to +\infty),$$

then $p = \omega \exp \mu[\omega = \exp(-\mu(R^n))]$ belongs to $I_\alpha^{\,n}$.

In the case $n = 1$, this theorem gives the following result stated by Fryntov [3]:

COROLLARY 6. *Let $a > 0$ and $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ where $\lambda_1 > 2a$ and λ_{k+1}/λ_k is an integer greater than one. If μ is a bounded univariate measure concentrated on $[a, 2a] \cup \Lambda$ and if for some $K > 0$*

$$\mu(\{x \in R : x > u\}) = O(\exp(-Ku^2)), \qquad (u \to +\infty),$$

then $p = \omega \exp \mu[\omega = \exp(-\mu(R))]$ belongs to $I_\alpha^{\,1}$.

5. Case of Enumerable Sets

We say that an enumerable set B has for basis $A = \{a_j\}$ if A is an independent set and if any point $x \in B$ can be written as

$$x = \sum_j x_j a_j , \qquad x_j \in Q,$$

the x_j being null except for a finite number.

Lemma 4. *Let B be an enumerable set with basis $A = \{a_j\}$ and μ be a bounded measure concentrated on B and satisfying for some $K > 0$ the estimation*

$$\mu\left(\left\{\sum_j x_j a_j\right\}\right) = O\left[\exp\left(-K \sum_j | x_j |\right)\right]$$

when $\sum_j | x_j | \to +\infty$. If $p = \omega \exp \mu[\omega = \exp(-\mu(R^n))]$, p_j is an n-variate probability and α_j is a positive constant ($j = 1,\ldots, m$) satisfying (2), *there exist some $b_j \in R^n$ and some $\nu_j \in \mathscr{M}_n$ satisfying the following conditions*:

(a) $\sum_{j=1}^m \alpha_j b_j = 0$;

(b) $p_j = \omega_j(\delta_{b_j} * \exp \nu_j)$, $\omega_j = \exp(-\nu_j(R^n))$;

(c) ν_j *is concentrated on* $(Z)B$ ($j = 1,\ldots, m$).

Proof. Since p is concentrated on $(Z)B$, it follows from [2, Theorem 9.1.1] the existence of some $b_j \in R^n$ such that $q_j = p_j * \delta_{-b_j}$ is concentrated on $(Z)B$. If r and r_j are the symmetrized of p and p_j, respectively

$$\hat{r}(t) = | \hat{p}(t)|^2, \qquad \hat{r}_j(t) = | \hat{p}_j(t)|^2,$$

r and r_j are concentrated on $(Z)B$. Since

$$\inf_{t \in R^n} \hat{r}_j(t) > 0,$$

from [2, Theorem 7.3.2], we deduce that

$$r = \omega^2 \exp \rho, \qquad r_j = \omega_j' \exp \rho_j$$

where ρ and ρ_j are concentrated on $(Z)B$ and satisfy

$$\sum_{j=1}^m \alpha_j \rho_j = \rho = \mu + \tilde{\mu}.$$

Let $A' \subset [1, 2[$ be an independent set equipotent to A, ϕ be a bijection of A

on A' and Φ be the isomorphism of $\mathscr{M}((Q)A)$ on $\mathscr{M}((Q)A')$ induced by ϕ. Then

$$\Phi(r) = \omega^2 \exp(\Phi(\rho)), \qquad \Phi(r_j) = \omega_j' \exp(\Phi(\rho_j)).$$

Since

$$\sum_{j=1}^{m} \alpha_j \Phi(\rho_j) = \Phi(\rho),$$

we have

$$\prod_{j=1}^{m} (\widehat{\Phi(r_j)})^{\beta_j} = \widehat{\Phi(r)}.$$

From the assumptions, $\widehat{\Phi(r)}$ is an analytic function without zeros in $R^n + i\Gamma$ where $\Gamma = \{y \in R^n: \|y\| < d\}$ for some $d > 0$ and the same holds for $\widehat{\Phi(r_j)}$ ($j = 1,\ldots, m$). From [2, Theorem 7.3.3], $\int e^{-(y,x)} \mid \Phi(\rho_j)\mid (dx)$ converges for any $y \in R^n$, $\|y\| < d$. Since

$$\Phi(q_j) * \Phi(\tilde{q}_j) = \Phi(r_j),$$

it follows from [2, Theorem 8.1.1] the existence of a signed measure ν_j' concentrated on $\phi((Z)B)$ such that

$$\Phi(q_j) = \omega_j \exp \nu_j'$$

where $\omega_j = \exp(-\nu_j'(R^n))$. If we let $\nu_j = \Phi^{-1}(\nu_j')$, then

$$q_j = \omega_j \exp \nu_j$$

and (b) and (c) are satisfied. From the uniqueness of De Finetti representation, we deduce that (a) is also satisfied.

From this Lemma, we deduce the

THEOREM 6. *Let A be an enumerable independent set. If $p = \omega \exp \mu$ where μ is a bounded measure concentrated on A and $\omega = \exp(-\mu(R^n))$, then p belongs to I_α^n.*

Proof. Let p_j be an n-variate probability and α_j be a positive constant ($j = 1,\ldots, m$) satisfying (2). From the preceding lemma, we deduce the existence of some $b_j \in R^n$ and $\nu_j \in \mathscr{M}_n$ such that $q_j = p_j * \delta_{-b_j} = \omega_j \exp \nu_j$ where ν_j is concentrated on $(Z)A$, $\omega_j = \exp(-\nu_j(R^n))$ and

$$\sum_{j=1}^{m} \alpha_j b_j = 0, \qquad \sum_{j=1}^{m} \alpha_j \nu_j = \mu.$$

Let $A' \subset [1, 2[$ be an independent set equipotent to A, ϕ a bijection of A on A', and Φ be the isomorphism of $\mathscr{M}((Z)A)$ on $\mathscr{M}((Z)A')$ induced by ϕ. From

$$\sum_{j=1}^{m} \alpha_j \Phi(\nu_j) = \Phi(\mu),$$

$$\Phi(p) = \omega \exp(\Phi(\mu)), \qquad \Phi(q_j) = \omega_j \exp(\Phi(\nu_j)),$$

we deduce that $\Phi(q_j)$ is an α-factor of $\Phi(p)$. Since, from Theorem 2, $\Phi(p)$ belongs to I_α^1, $\Phi(q_j)$ is infinitely divisible. Therefore q_j is also infinitely divisible and p belongs to I_α^n.

Likewise, we have the following extensions of [2, Theorems 8.3.1 and 8.3.2] (the proof is the same and can be omitted):

THEOREM 7. *Let Λ be a Linnik set constructed on an enumerable independent set $A = \{a_j\}$. If μ is a bounded measure concentrated on Λ such that each point of Λ has an issue with respect to μ and if for some $K > 0$*

$$\mu\left(\left\{x = \sum_{j=1}^{\infty} x_j a_j \in \Lambda : \sum_{j=1}^{\infty} x_j^2 > u^2\right\}\right) = O(\exp(-Ku^2)), \qquad (u \to +\infty),$$

then $p = \omega \exp \mu[\omega = \exp(-\mu(R^n))]$ belongs to I_α^n.

THEOREM 8. *Let $A = \{a_j\}$ be an enumerable independent set and $\Lambda \subset (Z)A$ be a Linnik set constructed on A. If μ is a bounded measure concentrated on Λ such that each point of Λ has an issue with respect to μ and such that*

$$\mu(\{x\}) = o\left[\exp\left(-2 \sum_{j=1}^{\infty} |x_j| \log(|x_j|)\right)\right]$$

for $x = \sum_{j=1}^{\infty} x_j a_j \in \Lambda$, $\|x\| \to +\infty$, then $p = \omega \exp \mu[\omega = \exp(-\mu(R^n))]$ belongs to I_α^n.

REFERENCES

[1] ČISTJAKOV, G. P. (1975). On a Dugué's problem (in Russian). *Dokl. Akad. Nauk SSSR* **224** 775–778.

[2] CUPPENS, R. (1975). *Decomposition of Multivariate Probabilities.* Academic Press, New York.

[3] FRYNTOV, O. E. (1974). On the α-factors of infinitely divisible laws (in Russian). *Mat. Fiz. Funk. Analysis* **5** 51–63.

[4] MASE, S. (1975). Decomposition of infinitely divisible characteristic functions with absolutely continuous Poisson spectral measures. *Ann. Inst. Statist. Math.* to appear.
[5] MASE, S. (1975). Decomposition of multivariate infinitely divisible characteristic functions with absolutely continuous Poisson spectral measures. *J. Multivariate Analysis* **5** 415–424.

Printed in Belgium